Monty Hall

We are presented with three doors, Door 1, Door 2, and Door 3. Monty Hall has put a prize behind one. The Contestant has to guess it, by choosing one. Suppose she chooses Door 1. Monty Hall, who knows the location of the prize and will not open that door, opens Door 3 and reveals that there is no prize behind it. He then asks the Contestant whether she wishes to change from her initial choice to Door 2. Will changing to Door 2 improve the Contestant chances of winning the prize? One may think that with two doors left unopened, the Contestant has 50 : 50 chances with either one, so there is no point for or against changing doors. However this is not true.

The Bayesian analysis

- The prize is behind Door i: D_i.

- We have $P(D_1) = P(D_2) = P(D_3) = 1/3$

- We call B the proposition "Monty Hall opens Door 3".

- It is assumed Monty Hall opens at random when he has a choice, hence $P(B) = 1/2$.

- The other probabilities are calculated as follows.

- When D_1, Monty Hall is free to choose Door 2 or Door 3, and thus $P(B/D_1) = 1/2$,

- When D_2 Monty Hall has to choose Door 3, and thus $P(B/D_2) = 1$

- When D_3 Monty Hall has to choose Door 2, and thus $P(B/D_3) = 0$

- Bayes' theorem

$$P(A/B) = (P(B/A) \times P(A))/(P(B)))$$

- Thus under the condition that the Contestant has chosen Door 1, we have:

- $P(D_1/B) = \dfrac{P(B/D_1) \times P(D_1)}{P(B)} = \dfrac{1}{3}$

- $P(D_2/B) = \dfrac{P(B/D_2) \times P(D_2)}{P(B)} = \dfrac{2}{3}$

- $P(D_3/B) = \dfrac{P(B/D_3) \times P(D_3)}{P(B)} = 0$

Dynamic epistemic logic

- MH's actions generates an epistemic model $EM_1 = (W_1, R_1^{MH}, R_1^C)$ where:

  W_1 : w_1 =car behind D_1, w_2 =car behind D_2, and w_3 =car behind D_3

  $R_1^{MH} := \{(w_1, w_1), (w_2, w_2), (w_3, w_3)\}$ and $R_1^C = W_1 \times W_1$.

- C's public action updates the epistemic model M_1 with the action model $AM_1 = (V_1, A_1^C, A_1^{MH})$ where

  $V_1 = \{a_1\}$ with a_1 : C chooses door D_1

  $A_1^C = \{(a_1, a_1)\}$

- The preconditions of a_1 guarantee it can be performed in any world of W_1.

- The result is a product model $PM_1 = M_1 \times A_1 = (T_1, S_1^C, S_1^{MH})$:

$T_1 = \{v_1, v_2, v_3\}$, with

$$\begin{array}{c} v_1 = (w_1, a_1) \\ v_2 = (w_2, a_1) \\ v_3 = (w_3, a_1) \end{array}$$

Given that $R_c^1 = W_1 \times W_1$ and $A_1^C a_1 a_1$,
it follows that $S_1^C = T_1 \times T_1$.

Dynamic epistemic logic continued

- Finally the product model $PM_1 = M_1 \times A_1$ is updated with the action model $AM_2 = (V_2, A_2^C, A_2^{MH})$ where

  $V_2 = \{a_2, a_3\}$ with a_2 : MH opens D_2 and a_3 : MH opens D_3

  $A_2^{MH} = \{(a_2, a_2), (a_3, a_3)\}$.

- Preconditions: a_2 can be performed in the possible worlds $v_1$and v_3; and a_3 can be performed in the possible worlds $v_1$and v_2.

- The result is a product model $PM_2 = PM_1 \times A_2 = (T_2, S_2^C, S_2^{MH})$ with T_2 consisting of four worlds

$$\begin{aligned} x &= (v_1, a_2) \\ y &= (v_1, a_3) \\ z &= (v_2, a_3) \\ u &= (v_3, a_2) \end{aligned}$$

and $S_2^C = \{(x,u),(u,x),(y,z),(z,y)\} \cup \{s : s \in T_2\}$.

Product updates and probabilities

- The agents' probabilities in product mode:

$$\begin{array}{ccc} v & \cdots & v' \\ \downarrow a & & \downarrow b \\ (v,a) & \cdots & (v',b) \end{array}$$

- We need to compute $P_{C,(v,a)}(v',b)$: the probability agent C assigns to the world (v',b) in the world (v,a).

- We need to know:

  - the probabilities $P_{C,v}(v')$ that C assigns to the world v' in v, and
  - the probabilities $P_{v'}(b)$ assigned to the action b in the world v'.

- the probabilities $P_{C,v}(u)$ for every u such that $R_C(v,u)$, and

- the probabilities $P_u(b)$.

- We compute $P_{C,(v,a)}(v,b)$ according to the formula:

$$P_{c,(v,a)}(v',b) \;\; = \;\; \frac{P_{c,v}(v') \times P_{v'}(b)}{\sum_{R_C(v,u)} P_{C,v}(u) \times P_u(b)}$$

We have

$$P_{c,(w_1,a_1)}(w_1,a_1) = \frac{1}{3}$$

and

$$P_{C,v_1}(v_2) = P_{C,(w_1,a_1)}(w_2,a_1) = \frac{1}{3}$$

- Finally

$$P_{c,(v_1,a_3)}(v_1,a_3) = \frac{P_{C,v_1}(v_1) \times P_{v_1}(a_3)}{P_{C,v_1}(v_1) \times P_{v_1}(a_3) + P_{C,v_1}(v_2) \times}$$

and

$$P_{c,(v_1,a_3)}(v_2,a_3) = \frac{P_{C,v_1}(v_2) \times P_{v_2}(a_3)}{P_{C,v_1}(v_1) \times P_{v_1}(a_3) + P_{C,v_1}(v_2) \times}$$

Monty Hall in IF logic: a zero-sum game of imperfect information

- A zero-sum game played by two players

- We focus on two kinds of strategies for player C.

- First kind: choose a door, then stick to it no matter what MH does.

- This strategy is encoded by three functions,$(D_1, h$ and (D_3, h_3), where

$$
\begin{aligned}
&h_1(D_1, D_2) = D_1,\ h_1(D_1, D_3) = D_1\\
&h_2(D_2, D_3) = D_2,\ h_2(D_2, D_1) = D_2\\
&h_3(D_3, D_2) = D_3,\ h_3(D_3, D_1) = D_3
\end{aligned}
$$

- Each of them wins in one case (when the the initial guess is correct) and looses in the other two.

Monty Hall in IF logic continued: C's strategies

- Second strategy: choose a door, and then after MH opens a door, switch your initial guess.

- It is encoded by three functions (D_1, f_1),(D_2, f_2), and (D_3, f_3) where

$$\begin{array}{l} f_1(D_1, D_2) = D_3,\ \ f_1(D_1, D_3) = D_2 \\ f_2(D_2, D_3) = D_1,\ \ f_2(D_2, D_1) = D_3 \\ f_3(D_3, D_2) = D_1,\ \ f_3(D_3, D_1) = D_2 \end{array}$$

- Each of them wins in two cases (when the initial choice is incorrect) and looses in one case (when the initial guess is correct).

Monty Hall in IF logic: Monty Hall's strategies

- We consider the strategy: "choose a door and put the prize behind it, and after C chooses a door, open any other door". It is encoded by 6 functions.

(D_1, g_1) $\quad g_1(D_1, D_1) = D_2$, $g_1(D_1, D_2) = D_3$, $g_1(D_1, D_3) = D_2$

(D_1, g_1^*) $\quad g_1^*(D_1, D_1) = D_3$, $g_1^*(D_1, D_2) = D_3$, $g_1^*(D_1, D_3) = D_2$

(D_2, g_2) $\quad g_2(D_2, D_1) = D_3$, $g_2(D_2, D_2) = D_1$, $g_2(D_2, D_3) = D_1$

(D_2, g_2^*) $\quad g_2^*(D_2, D_1) = D_3$, $g_2^*(D_2, D_2) = D_3$, $g_2^*(D_2, D_3) = D_1$

(D_3, g_3) $\quad$ $g_3(D_3, D_1) = D_2$, $g_3(D_3, D_2) = D_1$,
$g_3(D_3, D_3) = D_1$

(D_3, g_3^*) $\quad$ $g_3^*(D_3, D_1) = D_2$, $g_3^*(D_3, D_2) = D_1$,
$g_3^*(D_3, D_3) = D_2$

The strategic game

The following table registers the payoffs of the players for the strategies mentioned

	(D_1, g_1)	(D_1, g_1^*)	(D_2, g_2)	(D_2, g_2^*)	$(D_3,$
(D_1, h_1)	$(1,0)$	$(1,0)$	$(0,1)$	$(0,1)$	$(0,$
(D_2, h_2)	$(0,1)$	$(0,1)$	$(1,0)$	$(1,0)$	$(0,$
(D_3, h_3)	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(1,$
(D_1, f_1)	$(0,1)$	$(0,1)$	$(1,0)$	$(1,0)$	$(1,$
(D_2, f_2)	$(1,0)$	$(1,0)$	$(0,1)$	$(0,1)$	$(1,$
(D_3, f_3)	$(1,0)$	$(1,0)$	$(1,0)$	$(1,0)$	$(0,$

- Each of the strategies (D_i, h_i) is weakly dominated by a strategy (D_j, f_j).

- Hence this strategic game has the same value as the game whose payoffs are described in the following matrix:

The strategic game continued:

	(D_1, g_1)	(D_1, g_1^*)	(D_2, g_2)	(D_2, g_2^*)	$(D_3,$
(D_1, f_1)	$(0, 1)$	$(0, 1)$	$(1, 0)$	$(1, 0)$	$(1,$
(D_2, f_2)	$(1, 0)$	$(1, 0)$	$(0, 1)$	$(0, 1)$	$(1,$
(D_3, f_3)	$(1, 0)$	$(1, 0)$	$(1, 0)$	$(1, 0)$	$(0,$

- Let μ^* be the uniform probability distribution $\mu^*(D_i, f_i) = \frac{1}{3}$ and ν^* the uniform probability distribution $\nu^*(D_i, g_i) = \frac{1}{6}$ and $\nu^*(D_i, g_i^*) = \frac{1}{6}$.

- The pair (μ^*, ν^*) is an equilibrium

- The expected utility of player C for the pair (μ^*, ν^*) is $\frac{2}{3}$

Expressing Monty Hall in IF logic

- Monty Hall game is expressed in IF logic by the sentence

  $$\forall x(\exists y/\{x\})\forall z[x \neq z \wedge y \neq z \rightarrow (\exists t/\{x\})x = t]$$

or equivalently by the sentence φ_{MH}

$$\forall x(\exists y/\{x\})\forall z[x = z \vee y = z \vee (\exists t/\{x\})x = t]$$

- For each $i = 1, 2, 3$:

$$U_i((D_i, f_i), \nu) \;=\; \sum_{\tau \in S_\forall} \nu(\tau) u_i((D_i, f_i), \tau) = \frac{2}{3}$$

- On the other side, for each $i = 1, 2, 3$:

  $$U_i((D_i, h_i), \nu) = \sum_{\tau \in S_\forall} \nu(\tau) u_i((D_i, h_i), \tau) = \frac{1}{3}$$

Comparison: Two styles of logical analysis

- In DEL We are interested in the dynamics of information flow in the puzzle

- The agents are in a certain informational situation

- Their beliefs (including probabilistic beliefs about alternative situations) are revised or updated as a result of upcoming information

- DEL as a *logic *of* procedure*s is well suited to describe this step by step process

- Problem: The justification of prior probabilities

Comparison: IF logic, logic *as* procedure

- Reconceptualization of the puzzle as a game-theoretical, not a decision theoretical one

- As a consequence, Monty Hall becomes a full fledged player

- The solution is semantical: the Monty Hall game is expressed by a sentence

- The strategies of the players are somehow determined by logic (Skolem form and Kreisel form of a given statement)

- There is no need for prior probabilities

Test note.

A time marker:

Don't talk more than .